SLIM EXCEPTIONAL SETS IN WARING'S PROBLEM: ONE SQUARE AND FIVE CUBES

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1. Introduction. Amongst the most irritating open problems of Waring type is that of establishing the expected asymptotic formula for the number of representations of an integer as the sum of five cubes and a square of natural numbers. The technology currently available to practitioners of the Hardy-Littlewood method fails, by only the narrowest of margins, to deliver the sought-after conclusion, and indeed Vaughan [5] has succeeded in establishing a lower bound for the desired number of representations that misses that expected by only a positive constant factor. The purpose of this note is to demonstrate that, although the expected asymptotic formula may occasionally fail to hold, the set of such exceptional instances is extremely sparse.

In order to describe our conclusion, we require some notation. Let n be a large positive number, and denote by $R(n)$ the number of representations of n in the form

$$n = x_1^3 + x_2^3 + \cdots + x_5^3 + y^2, \tag{1}$$

with $y \in \mathbb{N}$ and $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 5)$. Also, let $\mathfrak{S}(n)$ denote the singular series associated with the additive representation (1), so that

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-6} S_3(q,a)^5 S_2(q,a) e(-na/q), \tag{2}$$

where we write $e(z)$ for $\exp(2\pi i z)$, and define

$$S_k(q,a) = \sum_{r=1}^{q} e(ar^k/q) \quad (k = 2, 3). \tag{3}$$

Then it is conjectured that

$$R(n) \sim \frac{\Gamma(3/2)\Gamma(4/3)^5}{\Gamma(13/6)} \mathfrak{S}(n) n^{7/6}, \tag{4}$$

1991 *Mathematics Subject Classification.* 11P05, 11P55.

Key words and phrases. Waring's problem, exceptional sets.

*Supported in part by NSF grant DMS-9970440.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

and here it is worth noting that the standard theory establishes that $1 \ll \mathfrak{S}(n) \ll 1$ uniformly in n (see, for example, Chapter 4 of Vaughan [7]). In order to assess how frequently the formula (4) might fail, we define an associated exceptional set as follows. When $\psi(t)$ is a function of a positive variable t, denote by $E(N;\psi)$ the number of integers n with $1 \leqslant n \leqslant N$ for which

$$\left| R(n) - \frac{\Gamma(3/2)\Gamma(4/3)^5}{\Gamma(13/6)}\mathfrak{S}(n)n^{7/6}\right| > n^{7/6}\psi(n)^{-1}. \tag{5}$$

It is convenient here, and elsewhere, to refer to a function $\psi(t)$ as being a *function of uniform growth*, when $\psi(t)$ is a function of a positive variable t, increasing monotonically to infinity.

Theorem 1. *Suppose that $\psi(t)$ is a function of uniform growth with $\psi(t) = O(t^\delta)$, for some sufficiently small positive number δ. Then there is a positive number c satisfying the property that*

$$E(N;\psi) \ll \psi(N)^2 \exp(c \log N/\log\log N).$$

It follows, in particular, that for each positive number ε, the asymptotic formula (4) fails to hold for at most $O_\varepsilon(N^\varepsilon)$ of the integers n with $1 \leqslant n \leqslant N$. No comparable conclusion has been available hitherto, although a conventional application of Bessel's inequality would yield a similar conclusion with $E(N;\psi) \ll \psi(N)^2 N^{1/3+\varepsilon}$. We remark that the above cited conclusion of Vaughan [5] shows that $R(n) \gg n^{7/6}$ for all large integers n, and that Sinnadurai [4] had previously established the expected asymptotic formula for the number of representations of an integer as the sum of a square and six cubes of natural numbers.

Our proof of the above theorem is based on the methods introduced in our recent work on slim exceptional sets, the underlying ideas being clearly illustrated in Wooley [8] and [10]. For the moment, it suffices to comment that our methods avoid a conventional application of Bessel's inequality in favour of explicit control of an exponential sum over the exceptional set itself.

Throughout, the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. Also, we write $[z]$ for the largest integer not exceeding z.

The author is grateful to Professors Jörg Brüdern and Bob Vaughan for stimulating conversations and suggestions on the topic of this paper.

2. Initial salvos. Our proof of Theorem 1 employs the Hardy-Littlewood method, and so we must introduce some notation before launching our argument in earnest. Let N be a large positive number, and let $\psi = \psi(t)$ be a function of the type described in the statement of Theorem 1. We denote by $\mathcal{Z}(N)$ the set of integers n with $N/2 < n \leqslant N$ for which the inequality (5) holds, and we abbreviate $\text{card}(\mathcal{Z}(N))$ to Z. For $k = 2, 3$, we write $P_k = [N^{1/k}]$ and define

$$f_k(\alpha) = \sum_{1 \leqslant x \leqslant P_k} e(\alpha x^k).$$

Then by orthogonality, for each integer n with $N/2 < n \leqslant N$, one has

$$R(n) = \int_0^1 f_2(\alpha) f_3(\alpha)^5 e(-n\alpha)\,d\alpha. \tag{6}$$

Next we define a general Hardy-Littlewood dissection employed in our application of the circle method. When X is a positive number with $X \leqslant \sqrt{N}$, we take $\mathfrak{N}(X)$ to be the union of the intervals

$$\mathfrak{N}(q, a; X) = \{\alpha \in [0, 1) : |q\alpha - a| \leqslant XN^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant X$ and $(a, q) = 1$. Also, when $X \leqslant \sqrt{N}/2$, we put $\mathfrak{K}(X) = \mathfrak{N}(2X) \setminus \mathfrak{N}(X)$. Finally, we take $\nu = 1/100$, write $\mathfrak{M} = \mathfrak{N}(N^\nu)$, and then set $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$.

It follows from the methods of Chapters 2 and 4 of Vaughan [7] that whenever $N/2 < n \leqslant N$, one has

$$\int_{\mathfrak{M}} f_2(\alpha) f_3(\alpha)^5 e(-n\alpha)\,d\alpha = \frac{\Gamma(3/2)\Gamma(4/3)^5}{\Gamma(13/6)} \mathfrak{S}(n) n^{7/6} + O(n^{7/6-2\delta}), \tag{7}$$

where $\mathfrak{S}(n)$ denotes the singular series defined in (2). Note here our use of the implicit assumption that δ is a sufficiently small positive number. But for $n \in \mathcal{Z}(N)$, it follows from (6), (7), and our assumed upper bound $\psi(t) = O(t^\delta)$, that

$$\Big| \int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha)^5 e(-n\alpha)\,d\alpha \Big| > \tfrac{1}{2} n^{7/6} \psi(n)^{-1}. \tag{8}$$

Define the complex number η_n by taking $\eta_n = 0$ for $n \notin \mathcal{Z}(N)$, and when $n \in \mathcal{Z}(N)$ by means of the equation

$$\Big| \int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha)^5 e(-n\alpha)\,d\alpha \Big| = \eta_n \int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha)^5 e(-n\alpha)\,d\alpha.$$

Plainly, one has $|\eta_n| = 1$ whenever η_n is non-zero. Then it follows from (8) that

$$\begin{aligned} N^{7/6}\psi(N)^{-1}\operatorname{card}(\mathcal{Z}(N)) &\ll \sum_{N/2<n\leqslant N} \eta_n \int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha)^5 e(-n\alpha)\,d\alpha \\ &= \int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha)^5 K(-\alpha)\,d\alpha, \end{aligned} \tag{9}$$

where the exponential sum $K(\alpha)$ is defined by

$$K(\alpha) = \sum_{N/2<n\leqslant N} \eta_n e(n\alpha).$$

Our strategy is to estimate the integral on the right hand side of the relation (9), and thereby obtain an upper bound for Z. This we achieve by performing what, technically speaking, amounts to a sequence of pruning procedures. None of the latter will prove demanding for experts in the circle method.

3. Pruning procedures. Before embarking on the first pruning process, we require some additional notation. We define the function $f_2^*(\alpha)$ for $\alpha \in [0,1)$ by taking

$$f_2^*(\alpha) = \sqrt{N \log N}(q + N|q\alpha - a|)^{-1/2}, \tag{10}$$

when $\alpha \in \mathfrak{N}(q, a; \sqrt{N}) \subseteq \mathfrak{N}(\sqrt{N})$. If there is ambiguity in the choice for q and a satisfying the latter condition, then we simply make the choice that maximises the right hand side of (10). Notice here that by Dirichlet's approximation theorem, whenever $\alpha \in [0,1)$, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $\alpha \in \mathfrak{N}(q, a; \sqrt{N})$, whence $[0,1) = \mathfrak{N}(\sqrt{N})$.

Lemma 1. *Uniformly for $\alpha \in [0,1)$, one has $f_2(\alpha) \ll f_2^*(\alpha)$.*

Proof. An inspection of the proof of Weyl's inequality for quadratic exponential sums reveals that the following upper bound holds (see, for example, the proofs of Lemmata 2.2 and 2.4 of Vaughan [7]). That is, when $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy $(a, q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$, then

$$f_2(\alpha) \ll P_2 \sqrt{\log(2qP_2)}(q^{-1} + P_2^{-1} + qP_2^{-2})^{1/2}. \tag{11}$$

Observe that the latter upper bound is trivial when $q \geqslant P_2^2$, and thus we may assume without loss of generality that $q \leqslant N$. But a standard transference argument (see, for example, Exercise 2 of Chapter 2 of Vaughan [7]) leads from (11) to the estimate

$$f_2(\alpha) \ll \sqrt{N \log N} \left((q + N|q\alpha - a|)^{-1} + N^{-1/2} + (q + N|q\alpha - a|)/N \right)^{1/2}. \tag{12}$$

Observe next that whenever $1 \leqslant X \leqslant \sqrt{N}$ and $\alpha \in \mathfrak{N}(q, a; X) \subseteq \mathfrak{N}(X)$, then one has $1 \leqslant q \leqslant X$, $(a, q) = 1$ and $|\alpha - a/q| \leqslant q^{-1}XN^{-1} \leqslant q^{-2}$. Thus the hypothesis required to obtain (12) holds, and furthermore one has $q + N|q\alpha - a| \leqslant 2X$, so that

$$(q + N|q\alpha - a|)^{1/2}\sqrt{\log N} \leqslant \sqrt{2X \log N} \ll N^{1/4}\sqrt{\log N},$$

and also

$$\sqrt{N \log N}(q + N|q\alpha - a|)^{-1/2} \geqslant (2X)^{-1/2}\sqrt{N \log N} \gg N^{1/4}\sqrt{\log N}.$$

We therefore conclude from (10) and (12) that whenever $1 \leqslant X \leqslant \sqrt{N}/2$ and $\alpha \in \mathfrak{K}(X)$, then

$$f_2(\alpha) \ll \sqrt{N \log N}(q + N|q\alpha - a|)^{-1/2} = f_2^*(\alpha).$$

The conclusion of the lemma consequently follows from our earlier observation that $[0,1) = \mathfrak{N}(\sqrt{N})$.

Before proceeding further, we require an approximation to $f_3(\alpha)$ valid uniformly for $\alpha \in \mathfrak{N}(\sqrt{N})$. Write

$$v_3(\beta) = \int_0^{P_3} e(\beta\gamma^3)d\gamma,$$

and recall the notation introduced in (3). We define the function $f_3^*(\alpha)$ for $\alpha \in [0,1)$ by setting

$$f_3^*(\alpha) = q^{-1}S_3(q,a)v_3(\alpha - a/q), \tag{13}$$

when $\alpha \in \mathfrak{N}(q,a;\sqrt{N}) \subseteq \mathfrak{N}(\sqrt{N})$. On this occasion, if there is ambiguity concerning the choice of q and a satisfying the latter condition, then we make a choice that maximises the right hand side of (10). Write $L = \exp(\log N/\log\log N)$. Then it follows from the argument of the proof of Theorem 4.1 of Vaughan [7] that there is a positive number c with the property that whenever $\alpha \in \mathfrak{N}(q,a;\sqrt{N}) \subseteq \mathfrak{N}(\sqrt{N})$, then one has

$$f_3(\alpha) - q^{-1}S_3(q,a)v_3(\alpha - a/q) \ll L^c(q + N|q\alpha - a|)^{1/2}. \tag{14}$$

In order to justify this bound, one must note that factors of q^ε that occur in the above cited argument originate as functions of q bounded as powers of $d(q)$, the number of divisors of q. Since for $\alpha \in \mathfrak{N}(q,a;\sqrt{N}) \subseteq \mathfrak{N}(\sqrt{N})$, one has $q \leqslant \sqrt{N}$, it follows that these factors of q^ε may be replaced here by a suitable fixed power of L, as recorded in (14).

It is convenient at this point to introduce the mean values

$$I_0 = \int_{\mathfrak{m}} |f_2(\alpha)f_3(\alpha)^5K(\alpha)|d\alpha \tag{15}$$

and

$$I_1 = \int_{\mathfrak{m}} |f_2^*(\alpha)f_3^*(\alpha)^2f_3(\alpha)^3K(\alpha)|d\alpha. \tag{16}$$

Lemma 2. *For each positive number ε, one has*

$$I_0 \ll I_1 + N^{7/6}L^{2c}Z^{1/2} + N^{41/36+\varepsilon}Z.$$

Proof. Observe first that, in view of Lemma 1, one has

$$I_0 \ll \int_{\mathfrak{m}} |f_2^*(\alpha)f_3(\alpha)^5K(\alpha)|d\alpha.$$

But by (10), (13) and (14), whenever $1 \leqslant X \leqslant \sqrt{N}/2$ and $\alpha \in \mathfrak{K}(X)$, one has

$$f_2^*(\alpha)f_3(\alpha) \ll |f_2^*(\alpha)f_3^*(\alpha)| + L^c\sqrt{N\log N}, \tag{17}$$

whence

$$I_0 \ll J_1 + L^c\sqrt{N\log N}J_2, \tag{18}$$

where

$$J_1=\int_{\mathfrak m}|f_2^*(\alpha)f_3^*(\alpha)f_3(\alpha)^4K(\alpha)|d\alpha \tag{19}$$

and

$$J_2=\int_{\mathfrak m}|f_3(\alpha)^4K(\alpha)|d\alpha. \tag{20}$$

By applying Schwarz's inequality to (20), one finds that $J_2\leqslant (J_3J_4)^{1/2}$, where

$$J_3=\int_0^1|f_3(\alpha)|^4d\alpha \quad \text{and} \quad J_4=\int_0^1|f_3(\alpha)^4K(\alpha)^2|d\alpha.$$

But on considering the underlying diophantine equation, it follows from Hooley [1] that $J_3\ll P_3^2$. Also, on employing Lemma 2.1 of Parsell [3] (see also Hooley [2]) in order to verify Hypothesis $\mathcal{R}(11/6)$ of Wooley [9], it follows from the argument of the proof of Lemma 10.3 of the latter paper that

$$J_4\ll ZP_3^2+Z^2P_3^{11/6+\varepsilon}.$$

We therefore conclude that

$$\begin{aligned}J_2&\ll P_3(ZP_3^2+Z^2P_3^{11/6+\varepsilon})^{1/2}\\ &\ll Z^{1/2}N^{2/3}+ZN^{23/36+\varepsilon}.\end{aligned} \tag{21}$$

Next, applying the upper bound (17) within (19), and recalling the definition (16), we see that

$$J_1\ll I_1+L^c\sqrt{N\log N}J_5, \tag{22}$$

where

$$J_5=\int_{\mathfrak m}|f_3^*(\alpha)f_3(\alpha)^3K(\alpha)|d\alpha.$$

An application of Hölder's inequality yields the upper bound

$$J_5\leqslant J_3^{1/6}J_4^{1/2}J_6^{1/6}J_7^{1/6},$$

where

$$J_6=\int_0^1|f_3(\alpha)|^2d\alpha \quad \text{and} \quad J_7=\int_0^1|f_3^*(\alpha)|^6d\alpha.$$

But Parseval's identity yields the estimate $J_6\leqslant P_3$, and the argument of the proof of Lemma 5.1 of Vaughan [6] shows that whenever $s\geqslant 5$, one has

$$\int_0^1|f_3^*(\alpha)|^sd\alpha\ll P_3^{s-3}.$$

Thus we conclude that $J_7 \ll P_3^3$, whence our earlier discussion leads to the bound

$$\begin{aligned} J_5 &\ll (P_3^2)^{1/6}(ZP_3^2 + Z^2P_3^{11/6+\varepsilon})^{1/2}(P_3)^{1/6}(P_3^3)^{1/6} \\ &\ll Z^{1/2}N^{2/3} + ZN^{23/36+\varepsilon}. \end{aligned} \tag{23}$$

Finally, on substituting (23) into (22), and then substituting the ensuing bound together with (21) into (18), we arrive at the upper bound

$$I_0 \ll I_1 + Z^{1/2}N^{7/6}L^c\sqrt{\log N} + N^{41/36+2\varepsilon}Z.$$

The conclusion of the lemma therefore follows whenever N is sufficiently large, as we may assume.

Having replaced two of the exponential sums $f_3(\alpha)$ by their well-behaved approximations $f_3^*(\alpha)$, the integral I_1 may already be estimated directly. We summarise the conclusion of this discussion in the following lemma.

Lemma 3. *One has* $I_1 \ll ZN^{7/6-\nu/9}$.

Proof. On applying Hölder's inequality to (16), we obtain

$$I_1 \leqslant K(0)J_8^{1/4}J_9^{3/8}J_{10}^{3/8}, \tag{24}$$

where

$$J_8 = \int_0^1 f_2^*(\alpha)^4 d\alpha, \quad J_9 = \int_{\mathfrak{m}} |f_3^*(\alpha)|^{16/3} d\alpha, \quad J_{10} = \int_0^1 |f_3(\alpha)|^8 d\alpha.$$

But Hua's lemma establishes that $J_{10} \ll P_3^{5+\varepsilon}$ (see, for example, Lemma 2.5 of Vaughan [7]), and on recalling (10), a direct calculation yields

$$\begin{aligned} J_8 &\leqslant (N\log N)^2 \sum_{1\leqslant q\leqslant\sqrt{N}} \sum_{a=1}^{q} \int_{-\infty}^{\infty} (q + qN|\beta|)^{-2} d\beta \\ &\ll N(\log N)^2 \sum_{1\leqslant q\leqslant\sqrt{N}} q^{-1} \ll N(\log N)^3. \end{aligned}$$

In order to dispose of J_9, we note that the argument of the proof of Lemma 5.1 of Vaughan [6] shows that whenever $s \geqslant 5$ and $1 \leqslant X \leqslant \sqrt{N}/2$, one has

$$\int_{\mathfrak{K}(X)} |f_3^*(\alpha)|^s d\alpha \ll P_3^{s-3}X^{\varepsilon-1/3}.$$

Then since

$$\mathfrak{m} \subseteq \bigcup_{\substack{i\geqslant 1 \\ 2^{1-i}\sqrt{N}\geqslant N^{\nu}}} \mathfrak{K}(2^{-i}\sqrt{N}),$$

we see that

$$\int_{\mathfrak{m}} |f_3^*(\alpha)|^{16/3} d\alpha \ll P_3^{7/3}(N^{\nu})^{\varepsilon-1/3}.$$

On substituting the above estimates into (24), we conclude that

$$I_1 \ll Z(N(\log N)^3)^{1/4}(N^{7/9-\nu/3+\varepsilon})^{3/8}(N^{5/3+\varepsilon})^{3/8},$$

and the conclusion of the lemma follows immediately.

4. The coup de grâce. The time has come to deliver the mortal blow. From the relation (9) and the definition (15), on the one hand, and Lemmata 2 and 3, on the other, we obtain the inequality

$$N^{7/6}\psi(N)^{-1}Z \ll I_0 \ll N^{7/6}L^{2c}Z^{1/2} + N^{7/6-\nu/9}Z.$$

Thus, whenever $\psi(t)$ is a function of uniform growth with $\psi(t) = o(t^{\nu/9})$, it follows that

$$Z \ll \psi(N)L^{2c}Z^{1/2},$$

whence $Z \ll \psi(N)^2 L^{4c}$. On summing over dyadic intervals, we conclude that

$$E(N;\psi) \ll \psi(N)^2 L^{4c} \log N \ll \psi(N)^2 L^{5c},$$

and this suffices to establish the conclusion of Theorem 1.

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